作者姓名：梁兴

论文题目：不动点指标理论及其在 K 型单调和竞争动力系统中的应用

作者简介：梁兴，男，1976 年 7 月出生，2001 年 9 月师从于中国科学技术大学蒋继发教授，于 2003 年 7 月获博士学位。

中　文　摘　要

很多类型的微分方程（组）的解都保持某种由初值条件、边值条件和（或）非齐次项生成的序关系。很多情况下，当我们求解、构造收敛于平衡点的单调序列或给出解的上下界时，我们都要利用这种保序性。基于单调性和比较原理的方法很长时期以来一直得到广泛应用，它的本质思想在 20 世纪初就已经被提及， 例如， Courant 和 Hilbert 1937 年提到 Bieberbach 1912 年描述的一种利用单调叠代解非线性椭圆方程的方法。在上世纪 20 到 30 年代，Hopf 和 Kamke 分别发展了偏微分方程的强极大值原理和常微分方程组比较原理。在此以后几十年中，单调方法成为微分方程的重要方法，并被抽象为正算子理论，作为泛函分析的一个重要分支得到充分发展。但是，上世纪 80 年代以前，大多数文献利用单调方法得到的结果往往是局部的。

上个世纪数学的另一重要发展则是对非线性发展方程（与时间相关的常微分方程、偏微分方程及泛函微分方程）的长期大范围性态的定性研究。Poincare 和 Lyapunov 是这一领域的开创者。经过几代数学家包括 Birkhoff、van Neumann、Kolmogorov 等人的努力，在整个 20 世纪里，这一领域持续、蓬勃地发展，并始终保持旺盛的活力，最终形成动力系统这一现代数学的重要分支，并且有了更深更广的数学内涵和更加广阔的应用。

特别是在上个世纪 60 到 70 年代，微分动力系统领域出现了一大批深刻的结果。由于偏微分方程理论和算子半群理论的应用，这些结果被进一步推广到无穷维的情况，并反过来促进了偏微分方程理论的发展。

当动力系统的观点和方法被引入后，单调性、正性和比较原理不但可以用来解决局部的、线性的或静态的数学问题，而且可以处理全局的、非线性的和动态的数学问题。在很多数学分支中零散的结论也可以被统一、抽象并提高。单调动力系统理论这一数学分支正是在动力系统和单调性思想的交叉、结合中产生的。

正如我们上面所介绍的，动力系统和单调性理论的发展已经为单调动力系统的产生准备好了必要的理论工具，而促使单调动力系统理论产生的另外一个因素则是实际应用的需要。从 20 世纪初起，在化学和生物领域的研究当中，建立了大量的数学模型，特别是微分方程模型。和传统的来自于力学和物理领域的模型不同，守恒律、对称性分析和变分原理在研究这些模型时一般不再适用。但是，在化学和生物领域的研究中，我们通常处理种群密度、化学物质浓度等物理量，它们当然都具有正值，因此，方程的解都保持正性。另一方面，这些模型还保持附加的单调性或保序性。

20 世纪 80 年代，Hirsch 和 Matano 各自独立地创建了单调动力系统理论。很多优秀数学家，如 Dancer、Hess、Polacik 和 Smith 对单调系统理论均做出了重要贡献，使这个领域在过去 20 多年中得到非常迅速的发展，取得了大量的深刻而具有重要意义的成果。这一领域的最大特点是不同分支问题的研究在方法论上的***一致性***和结果的***整体性***。如果研究者能够洞察问题的本质，则能够组织一个或若干个抽象结果， 使得各类具有比较原理的演化方程都能够应用。这篇博士论文将继承和发展这一特点。主要贡献包括以下四部分内容。

(C1) 从大量的实际模型抽象出 K 型单调映射的概念。研究了乘积 Banach 空间中正锥上 K 型单调映射的不动点指标理论，讨论了不动点指标与不动点局部稳定性之间的关系，给出了不动点指标和公式。进一步,把相应的结果推广到 Kolmogorov 微分方程系统，给出这类系统的平衡点指标。结合单调动力系统理论和不动点指标理论，研究了无穷维 K 型单调映射的持久性、全局收敛性及共存态的存在性、唯一性和全局吸引性。并把所得结果应用于时间周期的 Lotka－Volterra 反应扩散

系统，获得了与抽象结果相对应的结果，每一个结果均可以通过方程的系数判断。特别地，我们首次给出了正周期解全局稳定的可判性条件，解决了著名数学家，如 N. Dancer, P. Hess, Ahmad and Lazer 等人多次提出的这一“非常困难的问题”。当这些结果应用于 K 型单调的常微分方程系统，解决了 Smith（1986）提出的三个公开问题，并给其全局稳定性的研究划上了一个圆满的句号。这项工作分别发表在 J. Differential Equations 和 Trans. Amer. Math. Soc. 上.

(C2)我们研究了 K 型单调和竞争的自治 Kolmogorov 常微分系统的动力学性态，特别是几何约化性态。对于 K 型单调系统，我们证明了余维 1 的不变胞腔的存在性；对于 3 维 K 型单调系统，我们证明了系统的处处收敛性；结合平衡点指标理论，我们给出了 3 维 K 型单调的自治 Lotka-Volterra 系统动力学性态的完整分类。对于 K 型竞争系统，我们证明了存在至多可数个余维 1 的不变胞腔吸引了所有持久非收敛轨道，并给出条件保证所有的正轨道都被同一个这样流形的闭包所吸引;结合平衡点指标理论，我们给出了 3 维 K 型竞争的自治 Lotka-Volterra 系统动力学性态的分类，此分类完全由(两维)边界平衡点的稳定性态给出;进一步，我们证明此系统存在 Hopf 分支，并给出条件保证系统正平衡点的全局吸引性。我们还研究了迁移－竞争系统的动力学性态，并证明了：对四维迁移－竞争系统，Poincaré-Bendixson 定理成立。这项工作分别发表在 Nonlinearity 等杂志上.

(C3) 给出无限维动力系统的鞍点结构和广义鞍点结构的抽象定义，证明几个单调和 K 型单调动力系统的鞍点结构定理，由此解决了 Capasso-Wilson 猜测。我们的广义鞍点结构的抽象定义与 Smith 关于平衡点结构的猜测紧密相关。进一步，我们研究该猜测成立的最佳条件。一方面，在“稳定的平衡点是完全有序”的条件下，证明这一猜测成立；另一方面，用域扰动理论构造出一个反应扩散系统，说明：如果这一条件违背，则 Smith 的平衡点结构猜测不成立。这一工作已经发表在 J. Differential Equations 上。

(C4) 在抛物型方程行波解的研究中，绝大部分系统都具有极值原理。我们以抽象的观点看待行波解，把波形函数视为实数群或整数群到有序 Banach 空间的映射。由此发展了一套单调半流的渐近传播速度和抽象行波解的理论，使得各种各样具有比较原理的的演化系统的行波解问题在我们的理论框架下能够得到统一处理。对单稳定的情形，在适当的连续性和紧性假设下，我们给出了渐近传播速度和单调行波解的存在性，证明了渐近传播速度和最小波速相等，并给出渐近传播速度的估计。这项工作对著名数学家 Weinberger 的结果进行了非常不平凡的推广。抽象行波解的理论结果使得很多类型的方程（组），比如，标准的反应扩散方程（组）、（局部的和非局部的）柱体上的反应扩散方程、积分方程组、积分微分方程组、偏泛函微分方程组、（局部的和非局部的）格点方程等的问题可以被统一的解决。其中对柱体上的反应扩散方程的应用首次给出了这类方程的渐近传播速度，推广了著名数学家 Nirenberg 关于这类方程的工作。这项工作已被 Communications on Pure and Applied Mathematics 正式接受发表。

关键词：单调系统、反应扩散方程、Kolmogorov 系统、Lotka－Volterra 系统、不动点指标、行波解、渐近传播速度

Theory of Fixed Point Index and Its Applications to

Type-K Monotone and Competitive Systems

Liang Xing

ABSTRACT

The solutions of many kinds of differential equations preserve some type of order relation on initial data, boundary data, and/or inhomogeneous terms. People often use such order-preserving properties to find solutions or construct iteration schemes which converge monotonically to equilibria or provide bounds on other solutions. Methods of analysis based on monotonicity and comparisons have been around for a long time, and the essential ideas behind it were introduced around the beginning of the twentieth century. Courant and Hilbert referred to a 1912 note of Bieberbach describing a monotone iteration scheme for solving a nonlinear elliptic equation. In the 1920s and 1930s, Hopf established the strong maximum principle, and Muller and Kamke developed the comparison and monotonicity results for systems of ordinary differential equations. In the next decades, methods of monotonicity became a very important tool in the field of differential equations. Furthermore, it was abstracted as positive operator theory, one of main parts in functional analysis and got a steady and sufficient development. However, methods of monotonicity were often used in ad hoc ways to derive local results before the 1980s.

At the same time, the theory of dynamical systems, established by Poincaré, Lyapunov and Birkhoff, developed rapidly. Especially, from the 1960s through the 1970s, a lot of significant results in this field came forth. Such results are gradually extended to infinite-dimensional settings in some extend via the development of the theory for semigroup of operator,

The works in these two fields introduced above provided a steady basis for the application of monotonicity to the global dynamical behavior of the differential equations. Another major trend setting the stage for monotone dynamical systems theory is the requirement of other subjects in mathematics and science. In the study of biology and chemistry, a lot of differential equations models were established. It is different from the models for mechanics that for many of these models the traditional variational principles, symmetry arguments, and conservation laws of mathematical physics are not available. On the other hand, since biological and chemical models typically treat quantities such as population densities or concentrations of chemicals which are intrinsically positive, they usually preserve positivity of solutions and often have additional monotonicity or order-preserving properties. There was a surge of interest in such nonlinear equations, especially in reaction-diffusion equations, people wished to use a uniform method to study a big amount of these systems.

In the 1980s, Hirsch and Matano independently founded the theory of monotone dynamical systems. Many excellent mathematicians, such as Dancer, Hess, Polacik and Smith, threw themselves into this field and greatly contributed to it. Thus, this field developed rapidly and a lot of very deep and valuable works appeared. The biggest characteristics or advantages in this field are ***UNIFYING*** in methods in different categories and ***GLOBAL*** in behavior if one sees through clearly the essence of problems.

This thesis inherits and develops these characteristics. The major contributions of it are the following:

(**C1**) Abstracting the concept of the type-K monotone mapping from various equations. The relation between the index and the local stability of a fixed point for such a mapping is studied. Moreover, the global dynamical behavior of the mapping is made clear by using the formula of the indices sum. The basin of attraction for a stable fixed point is exactly obtained and the sufficient and necessary conditions for the global stability of a positive fixed point are given. When these results are applied to time-periodic type-K monotone partial differential equations (PDEs), a so-called "very difficult problem" (see N. Dancer, P. Hess and other's papers) about the global stability of the positive periodic solution is solved. When these results are applied to type-K monotone ordinary differential equations (ODEs), three open problems proposed by Smith in one paper in 1986 are solved. This means the end for studying about global behavior of such systems.

(**C2**) Studying the geometrical reduction for type-K monotone and competitive Kolmogorov ODEs. It is proved that the dynamical behavior of type-K monotone and competitive Kolmogorov ODEs is 1-codimensional, i.e., all trajectories with nontrivial limit behavior will approach an invariant 1-codimensional Lipschitz manifold, all these Lipschitz manifolds are at most countable in cardinality. For many well-known systems, such as Lotka-Volterra systems, this kind of manifold is unique. Hence, the dynamics of the 3-dimensional Lotka-Volterra systems can be completely classified.

(**C3**) Defining the concept for the (generalized) saddle point structure in infinite dimensional systems. Then proving the theorems for monotone or type-K monotone systems to possess this structure. The aim to abstract this concept is at two conjectures: one is so-called Capasso and Wilson's; the other is Smith's equilibria structure conjecture. The theorems give Capasso and Wilson's conjecture an affirmative answer and the best possibility for Smith's equilibria structure conjecture to hold.

(**C4**) Putting a new insight into traveling waves on various evolution systems and regarding the profiles of traveling waves as a mapping from real line or integer set to a suitable ordered Banach space. Under this viewpoint, the theory of asymptotic speeds of spread and monotone traveling waves is uniformly established within a rather extensive framework. The results can be applied to a large amount of evolution systems.

Key words：Monotone Systems，Reaction-Diffusion Equations，Kolmogorov Systems, Lotka－Volterra Systems, Index of Fixed Point, Traveling Waves, Asymptotic Speed of Spread