附件 6

作者姓名：徐岩

论文题目：非线性波动方程的间断有限元方法

作者简介：：徐岩，女， 1978 年 1 月出生，2003 年 2 月师从于中国科学技术大学舒其望教授，于 2005 年 7 月获博士学位。

中 文 摘 要

本文应用间断有限元方法对一系列的高阶导数非线性波动方程进行研究，这些非线性方程包括一维标量方程、一维方程组和二维标量方程，涉及到 KdV-Burgers 方程、Kuramoto-Sivashinsky 方程、五阶导数 KdV 方程、五阶导数完全非线性 K(n,n,n)方程、非线性 Schrödinger 方程、耦合 KdV 方程组、二维 Kadomtsev-Petviashvili 方程和二维 Zakharov-Kuznetsov 方程。

对于一维情形，我们所构造的格式推广了 Yan 和 Shu 及 Levy、Shu 和 Yan 关于间断有限元方法求解高阶导数偏微 分方程的工作。事实上，对于五阶导数 KdV 方程和耦合 KdV 方程组，现有数值求解方法的研究还不是很深入。五阶导数完全非线性 K(n,n,n)方程具有一类紧致结构的解，由于紧致结构解没有足够的光滑性，会产生高频的色散误差，因此设计稳定且具有高精度的数值格式具有很大的挑战性。

对于二维 Kadomtsev-Petviashvili 方程，其方程本身是非适定的，初值问题的解不唯一。这一问题的出现主要是由于 Kadomtsev-Petviashvili 描述了 $\partial_x u$ 的演化过程而非 u 自身。因此，某些关于 u 的限制对于保证解的唯一性和适定性是非常必要的。我们针对这一限制条件构造了一类新的间断有限元基函数，该基函数既具有间断有限元基函数的局部性质，又能保持 Kadomtsev -Petviashvili 方程中全局算子的性质。另外，我们还将间断 Galerkin 有限元方法应用于二维 Zakharov-Kuznetsov 方程，将该方法推广到求解二维非线性波动方程。

最后，我们对于几类非线性 Schrödinger 方程设计了间断 Galerkin 有限元方法。这些方程包括广义非线性 Schrödinger 方程、二维非线性 Schrödinger 方程和耦合非线性 Schrödinger 方程，这是间断有限元方法的一个新的应用领域。尽管 Karakashian 等也考虑了间断有限元方法，但实际上是指时间离散上的间断，与这里的空间离散上的间断是不同的。对于二维非线性 Schrödinger 方程具有奇异解的问题，我们得到了很好的数值模拟结果。

对于给出的求解格式我们证明了其非线性 L^2 稳定性，这些结果对于相当一般的非线性情形，任意空间维数的任意三角剖分上的任意阶数的多项式均成立，不需要任何的非线性限制器。对于 K(n,n,n)方程证明了 n 为奇数情形的 L^{n+1} 稳定性。并且对于线性情形的半离散格式给出其误差估计 k+1/2 阶的证明，并在数值上验证了对 P^k 多项式，在

均匀和非均匀网格下格式对 L^2 范数和 L^∞ 范数均能达到 k+1 阶精度。

在时间离散方面，我们将指数时间离散方法与间断有限元方法结合，使得格式在得到高精度的同时又具有良好的稳定性，从而避免了传统的 TVD Runge-Kutta 方法由于空间导数的增高而导致的时间步长苛刻的限制条件。我们的求解格式能够直接应用于周期和非周期边值问题。对于各种方程的求解格式进行大量的数值模拟试验，证明了间断有限元方法求解这些高阶导数非线性波动方程的优越性。

关键词：间断有限元方法， 非线性波动方程，指数时间离散，稳定性

Local Discontinuous Galerkin Methods for Nonlinear Wave Equations

Xu Yan

ABSTRACT

The aim of this work is to develop the local discontinuous Galerkin methods to solve nonlinear wave equations with high order derivatives. These nonlinear wave equations include one-dimensional scalar problems, one-dimensional systems and two-dimensional scalar problems. These problems concern KdV-Burgers (KdVB) type equations, Kuramoto-Sivashinsky (KS) type equations, fifth-order KdV type equations, fifth-order fully nonlinear K(n,n,n) equations, nonlinear Schrödinger (NLS) equation, Ito-type coupled KdV equations, Kadomtsev-Petviashvili (KP) equation and Zakharov-Kuznetsov (ZK) equation.

Our schemes for one-dimensional problems extend the previous work of Yan and Shu and of Levy, Shu and Yan on LDG methods solving partial differential equations with higher spatial derivatives. The fifth-order fully nonlinear K(n,n,n) equations support compacton solutions. The lack of smoothness at the edge of compacton introduces high-frequency dispersive errors into the calculation. It is a challenge to design stable and accurate numerical schemes for solving K(n,n,n) equations.

The KP equation is not well-posed because the KP equation describes the time evolution of ∂_xu, rather than u itself. Some global constraints on u are necessary to guarantee unique solution and well-posedness. Our proposed scheme for the KP equation is by the use of a new class of piecewise polynomial basis functions, which satisfies the constraint from the PDE containing a non-local operator and at the same time has the local property of the discontinuous Galerkin methods. The scheme is also easy for implementation. The scheme for the ZK equation extends the previous work on LDG methods solving one-dimensional nonlinear wave equations to the two-dimensional setting.

We design a class of LDG methods for the NLS equations. Although Karakashian and Makridakis also considered a discontinuous Galerkin method, it refers to a discontinuous Galerkin discretization in time, hence it is different from our approach of using a LDG discretization for the spatial variables. We can get good resolution for two-dimensional NLS equation with singular solutions.

The LDG methods for all the equations mentioned above have excellent provable nonlinear stability. We prove a strong L^2 stability for the square entropy, for quite general nonlinear cases, for any orders of accuracy on arbitrary triangulations in any space dimension, without the need for nonlinear limiters. Our proposed scheme for K(n,n,n) has L^{n+1} stability for odd n.

We perform a detailed theoretical and numerical investigation of the methods. For the semi-discrete LDG methods, we obtain error estimates of (k+1/2)-th order accuracy in the L^2 norm for the linearized problems. Numerically, we typically obtain (k+1)-th order in both L^2 norm and L^∞ norm for nonlinear problems.

We use the explicit exponential time differencing (ETD) method by Cox and Matthews and by Kassam and Trefethen, for the time discretization coupled with the LDG method. The discontinuous Galerkin method is originally coupled with explicit and nonlinearly stable high order Runge-Kutta time discretization by Shu and Osher. But the explicit stability limit decreases dramatically for the high derivatives problem. The ETD schemes can achieve high order accuracy and maintain good stability while avoiding the severe explicit stability time step limit of the traditional Runge-Kutta discontinuous Galerkin method, which use explicit and nonlinearly stable high order Runge-Kutta time discretization, due to the presence of the high order derivative terms.

The methods for all these nonlinear equations can be easily designed for periodic and non-periodic boundary conditions. We perform extensive one and two dimensional numerical experiments for nonlinear problems to demonstrate the accuracy and capability of the LDG methods.

Key words: local discontinuous Galerkin methods, nonlinear wave equations, exponential time differencing method, stability