附件 2

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论文题目：动力学性质的相对化与局部化

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中 文 摘 要

系统的复杂性是动力系统研究领域中的一个重要课题。在描述系统复杂性的语言中，熵的概念占有着基本的位置。1958 年 Kolmogorov 对$\mathbb{Z}$-作用下保测系统引入了测度熵；不久，1965 年 Adler、Konheim 和 McAndrew 对$\mathbb{Z}$-作用下拓扑动力系统引入了拓扑熵。随后，很多数学家关注于$\mathbb{Z}$-作用下测度熵与拓扑熵的内在关系，并最终建立了两者之间的变分原理。引入测度熵的同时，Kolmogorov 定义了一类十分重要的正熵系统——测度 Kolmogorov 系统（简称 K-系统）。K-系统的研究一直以来都是遍历理论的主要课题之一，Furstenberg、Katok、Ornstein、Rohlin、Rudolph、Sinai、Weiss 等著名数学家为此做出了杰出贡献。鉴于遍历理论与拓扑动力系统惊人的平行性，多年来人们一直寻求着引入 K-系统的拓扑对应。上世纪九十年代初，法国数学家 Blanchard 首先对此取得重大突破，通过局部化的观点合理地引入了$\mathbb{Z}$-作用下的拓扑 K-系统。随后，Glasner、Host、黄文、Rudolph、Weiss 和叶向东等学者投入到它的研究中，得到熵的局部变分原理，并逐步建立起$\mathbb{Z}$-作用下动力系统的局部熵理论。局部熵理论使得人们能够更好地理解和把握正熵系统的复杂性。目前，关于局部熵理论仍然有许多基本的问题有待人们去解决，其中最为核心的问题包括：对$\mathbb{Z}$-作用下动力系统的局部熵理论进行更为深入地研究，对 amenable 群作用下动力系统建立局部熵理论。

Amenable 群的类很广，包括所有的有限群，紧致群和可解群。群的 amenability 可以等价定义为：它在每个紧致 Hausdorff 空间上的作用一定存在着不变测度。相对于$\mathbb{Z}$-作用而言，amenable 群作用下动力系统与熵有关的研究进展较为缓慢和困难。Ornstein 和 Weiss 在 1987 年的文章，是 amenable 群作用下动力系统熵理论的里程碑式的工作，为进一步的研究奠定了坚实的基础，提供了基本的工具。随后，众多优秀的数学家致力于它的研究，并取得了一系列重要的成果。如：2000 年 Rudolph 和 Weiss 在 Ann. Math.上的文章中对可数离散 amenable 群作用证明了完全正熵蕴含所有序的混合；2001 年 Lindenstrauss 在 Invent. Math.上对局部紧致 amenable 群作用下保测系统证明了逐点遍历定理；2001 年 Danilenko 利用轨道的观点重新建立了可数离散 amenable 群作用的熵理论（Rudolph 在 2002 年国际数学家大会的 45 分钟特邀报告中专门汇报了这些工作）。即使如此，amenable 群作用下动力系统熵理论的研究中仍有很多重要问题有待解决，例如，2005 年 Katok 和 Dooley 在黄山举行的一个动力系统国际会议上提出：是否可以对可数离散 amenable 群作用建立局部熵理论？

描述系统复杂性的另外一个重要语言是混沌的概念。鉴于混沌现象在自然界中大量存在，混沌现象的研究不仅是现代动力系统学科最活跃的分支之一，也是当今非线性科学领域的主要课题之一。“混沌”一词最早由李天岩和Yorke在1975年提出。由于不同领域的专家对混沌的不同认识和理解，目前有着多种不同的混沌定义。各种混沌现象间的关联性是人们一直关心的基本问题。Blanchard、Glasner、Kolyada和Maass在2002年取得了本质进展，他们

利用遍历论的方法证明了正熵蕴含Li-Yorke混沌。遍历论方法介入混沌的研究为人们提供了新的观点和强有力的工具，使人们更为清晰地认识正熵系统中发生混沌现象成为可能；同时也产生了充满挑战的问题：是否正熵蕴含着某种比Li-Yorke混沌更强的混沌属性？如果是，这些混沌发生在何处？

本文的主要目的是：着眼于遍历理论与拓扑动力系统的平行之处，有机地结合局部与整体的思想，对动力学性质进行局部化和相对化，从而系统、全面地研究上述提出的几个问题。

对应于上述问题，本文的主体内容分为以下三部分：

一、$\mathbb{Z}$-作用的局部熵理论

在该部分中，我们进一步细化前人的局部化思想，从而更好地理解了正熵系统的复杂性。

1. 在拓扑和测度意义下同时引入熵集、熵序列的概念，建立了两种熵集、熵序列的变分关系，并证明了正熵的动力系统一定存在一个包含不可数个点的熵集。事实上，我们指出正熵系统的复杂性集中在系统的熵集上：动力系统熵集拓扑熵的上确界等于整个系统的拓扑熵。相关结果发表于Nonlinearity, 19 (2006), no. 1, 53-74。
2. 在拓扑和测度意义下同时引入熵点的概念，建立了两种熵点的变分关系，研究了它们的提升和投影性质。值得提出的是，熟知每个只包含可数个点的动力系统拓扑熵为零，而我们关于熵点研究的一个令人意外的副产品是：每个正熵的动力系统里一定具有一个可数闭子集，使得它的拓扑熵等于整个系统的拓扑熵。相关结果发表于Transactions of the American Mathematical Society, 359 (2007), no. 12, 6167-6186。该结果还为人们研究动力系统的复杂性提供了“分析子集拓扑熵”的新角度，沿着这个角度我们可以得到一些有意思的结果，这将在本摘要的末尾具体提及。

二、更广泛群作用的局部熵理论

在该部分中我们借鉴前人关于$\mathbb{Z}$-作用的局部化思想，并结合Ornstein和Weiss对amenable群作用建立起来的标准熵理论、Danilenko等人的轨道观点，对可数离散amenable群作用的局部熵理论首次进行系统、全面、深入地研究，特别的，我们肯定地回答了Katok和Dooley提出的问题。具体说来：我们对这类更广泛的动力系统建立了熵的局部变分原理；由此，我们对熵的概念进行局部化，在拓扑和测度意义下同时引入熵串的概念，建立两种熵串的变分关系；进一步，以熵串为出发点，我们引入并深入地研究了可数离散amenable群作用下K-系统的拓扑对应：n-一致正熵系统、完全一致正熵系统和完全正熵系统。这些结果覆盖了十几年来人们关于$\mathbb{Z}$-作用局部熵理论研究的几乎所有结果。Glasner评价我们的结果是“important and useful”。值得提出的是：在建立局部熵变分原理的过程中，对于有限Borel覆盖我们在测度意义下引入了两种熵，并证明了两者的等价性，由此肯定地回答了Kerr和Li在《combinatorial independence in measurable dynamics》一文中提出的Question 2.10（他们的论文可以从http://arxiv.org/abs/0705.3424v1下载）。为了证明两者的等价性，

1. 一方面，除了利用Ornstein和Weiss建立的标准熵理论，我们还必须借助于Connes、Danilenko、Dye、Feldman、Moore、Weiss、Zimmer等数学家发展起来的轨道理论。
2. 另一方面，作为桥梁，我们不得不研究相对化情形（即给定动力系统间的因子映射）下动力系统的局部熵理论。关于相对化情形下动力系统熵的局部变分原理的结果发表于Ergodic Theory and Dynamical Systems, 26 (2006), no. 1, 219-245，该结果被众多数学家引用，引用者包括丰德军、Glasner、H. Y. Hu、Kerr、H. F. Li、Shapira、易英飞等，并被Ergodic Theory and Dynamical Systems官方网站列为本期刊2007年人们下载阅读最多的10篇文章之一。在这个结果的证明过程中，我们发现在前人工作中起关

键作用的组合引理很难推广到相对化情形，因此我们需要利用新的完全不同于前人的方法来解决问题；同时，限制到绝对情形下的$\mathbb{Z}$-作用上，我们在文献中第一次指出有限Borel覆盖上述两种测度熵的等价性，这个结果在可数离散amenable群作用局部熵理论的建立过程中起到了不可替代的作用。作为应用，关于相对化情形下动力系统局部熵理论的系统研究的结果发表于Israel Journal of Mathematics, 158 (2007), 249-284，其中，我们自然地引入了相对拓扑Pinsker因子（对立于遍历理论中的Pinsker σ-代数），这回答了Park和Siemaszko提出的一个问题。

三、熵与混沌之间的关系

在该部分中，我们对相对化情形下动力系统的熵、混沌等复杂性进行了深入地研究，证明了：正的条件熵不仅蕴含了纤维上非平凡渐近对的存在性，还蕴含了相对Mycielski混沌；事实上，正条件熵情形下纤维上混沌集拓扑熵的上确界等于因子映射的相对拓扑熵。相关结果发表于Ergodic Theory and Dynamical Systems, 27 (2007), no. 4, 1349-1371和Journal of London Mathematical Society, 73 (2006), no. 1, 157-172，审稿人认为这些结果“significantly deeper than the corresponding absolute statements”、“opens a new view on topological entropy”、“the absolute version of Theorem 4.4 … is striking”。

攻博期间及获得博士学位后一年内，我和我的合作者还展开了如下研究：

1. 沿着分析子集拓扑熵的角度研究动力系统的复杂性。一个长时间的开问题是：是否每个拓扑动力系统都具有符号扩充？Boyle、Downarowicz、D. Fiebig、U. Fiebig、Glasner、Newhouse、Serafin、Weiss等人对此都进行过深入地研究。直到2004年Boyle和Downarowicz一道彻底地回答了这个问题，他们证明了：给定的拓扑动力系统具有principal的符号扩充当且仅当它为渐近h-扩张的。我们给出渐近h-扩张的拓扑动力系统一个全新的动力学刻画：给定的拓扑动力系统为渐近h-扩张的当且仅当它遗传一致可降。同时，我们还指出每个具有有限拓扑熵的系统一定可降。相关结果已被Ergodic Theory and Dynamical Systems接收，审稿人评价它为“well-worth publishing”。
2. 内蕴地引入（整体和局部）测度压的概念，给出它的等价刻画，并建立它与拓扑压之间的变分关系。相关结果已被Discrete and Continuous Dynamical Systems, Ser. A接收。
3. 研究与熵的局部变分原理相关的一些问题。通过研究，我和我的合作者发现：这些问题与很多数学分支都有着深刻的联系。
4. 混沌现象的广泛存在性及可观察性。我们致力于，在更加广泛的意义下研究动力系统的熵、混沌等复杂性，进而发现，一定条件下，混沌现象总是发生的，并且此时混沌集可以从很多种不同的角度被“观测”到。

关键词：局部熵、amenable 群、局部变分原理、条件熵、相对混沌

Relativization and Localization of Dynamical Properties

Zhang Guo Hua

ABSTRACT

The complexity of a system is a very important topic in the study of dynamical systems, and entropy is a fundamental concept to describe it. In 1958, Kolmogorov introduced measure-theoretic entropy for a measure-preserving system of $\mathbb{Z}$-actions; later, in 1965, Adler, Konheim and McAndrew introduced topological entropy for a topological dynamical system of $\mathbb{Z}$-actions. From then on, many people paid much attention to the study of the intrinsic relation between measure-theoretic entropy and topological entropy, and at last established the classical variational principle concerning entropy. With the introducing measure-theoretic entropy, Kolmogorov defined a very important class of measure-preserving systems with positive entropy, which is now known as the measure-theoretical Kolmogorov systems (K-systems, for short). The study of K-systems is one of the main research topics in ergodic theory, Furstenberg, Katok, Ornstein, Rohlin, Rudolph, Sinai and Weiss did outstanding contributions in this field. Observing the remarkable parallelism between ergodic theory and topological dynamics, it has drawn heavy attention for decades to seek the topological counterpart of a K-system. The first important breakthrough was made by Blanchard in 1992, who introduced topological K-systems of $\mathbb{Z}$-actions from the viewpoint of localization. Whereafter, many mathematicians, including Glasner, Host, W. Huang, Rudolph, Weiss and X. Ye, came into the study of the problem and established the so-called local entropy theory of $\mathbb{Z}$-actions, so that we could gain a better understanding of the complexity of a system with positive entropy. However, it remains many fundamental questions about the local entropy theory to be solved, especially, making a deeper study of the local entropy theory of $\mathbb{Z}$-actions, establishing the local entropy theory for an amenable group action, and so on.

The class of amenable groups is very large, including all finite groups, compact groups and solvable groups. An equivalent characterization of the amenability is that, there always exist invariant probability measures for any action of the group on a compact Hausdorff space. Comparing to $\mathbb{Z}$-actions, the level of development of entropy theory of amenable group actions lagged behind. However, this situation is rapidly changing in recent years. A turning point occurred with Ornstein and Weiss pioneering paper in 1987, which laid a foundation for an amenable group action. Henceforth, many mathematicians came into its study and obtained many important results. For example, in a paper on Ann. Math. Rudolph and Weiss proved in 2000 that, for the action of a countable discrete amenable group, completely positive entropy implies mixing of all orders; in a paper on Invent. Math. Lindenstrauss obtained in 2001 the pointwise ergodic theorem for general locally compact amenable group actions; in 2001 Danilenko developed an orbital approach to the entropy theory of countable discrete amenable group actions; and so on (Rudolph reported these results in his invited report on ICM 2002). Even so, there remain many important questions for general amenable group actions, for example, in 2005 Katok and Dooley proposed a question in an international conference of dynamical systems held in Huangshan: can we establish the local entropy theory for a countable discrete amenable group action?

Chaos is another useful concept in describing the complexity of a system. With the abounding of chaotic phenomena in the nature, the study of it is not only one of the most active branches in dynamical systems, but also one of the main research areas in nonlinear systems. The concept of chaos was introduced into mathematics first by T. Y. Li and Yorke in 1975. Because of a large amount of variant chaos, people have been concentrated on the study of the connection between different types of chaos for many years. In 2002, Blanchard, Glasner, Kolyada and Maass obtained an essential development by proving that a system with positive entropy must be Li-Yorke chaotic using the methods of ergodic theory. The intervention of ergodic theory into the study of chaotic phenomenon presents some new viewpoint and a powerful tool, which makes possible the existence of chaotic phenomena in a system with positive entropy. This also provides some challenging problem in the study of chaotic phenomenon: implied by the positivity of entropy, is there some chaotic property stronger than Li-Yorke chaos? If there is, where these chaos happen?

The main purpose of this dissertation is: starting from the remarkable similarities between ergodic theory and topological dynamics, we localize and relativize the dynamical properties combined with ideas of the local and the whole, and so conduct a systematical study of the problems mentioned as above.

According to these problems, our main results are divided into three parts as follows:

Part 1. the local entropy theory of $\mathbb{Z}$-actions.

In this part, following the idea of localizing entropy we obtained some results concerning the complexity of a system with positive entropy.

1. We introduced the concepts of entropy sets and entropy sequences both in measure-theoretical and topological situations, established the variational relation between these concepts in two settings, and proved the existence of an entropy set containing uncountably many points in each system with positive entropy. In fact, the entropy of a system is concentrated on the entropy set, precisely, the supremum of the entropies over all entropy sets is just topological entropy of the whole system. These results have been published on **Nonlinearity**, 10 (2006), no. 1, 53-74.
2. We introduced the concept of entropy points both in measure-theoretical and topological situations, established the variational relation between them, and studied the properties of lift-up and projection. It is well known that a system must have zero entropy if it contains just countably many points, a somewhat surprising consequence of our results is that there is a countable closed subset in each system whose entropy is equal to topological entropy of the original system. These results have been published on **Transactions of the American Mathematical Society**, 359 (2007), no. 12, 6167-6186. Note that, to describe the complexity of a system the study of entropy points shows us a new way of considering the entropy of subsets. In fact, following this line we can obtain many other interesting results which will be mentioned in details at the end of the abstract.

Part 2. the local entropy theory of a general group action.

In this part, combined with the standard machinery developed by Ornstein and Weiss and Danilenko's orbital approach, we followed the idea of localization and threw the first light on systematical studying of the local properties of entropy for a countable discrete amenable group action, in particular, we answered affirmatively the question of Katok and Dooley. Precisely, for a countable discrete amenable group action, we built the local variational principle concerning entropy, introduced the notion of entropy tuples in both topological and measure-theoretical situations and then established the variational relationship between these two kind of entropy tuples; last, based on the notion of entropy tuples, we studied the topological counterpart of a K-system: n-uniformly positive entropy system, completely uniform positive entropy system and completely positive entropy system. Our results covered almost all recent results in the local entropy theory of $\mathbb{Z}$-actions. Glasner commented our preprint with “important and useful”. Note that, in the building of the local variational principle, we introduced two kind of measure-theoretical entropy of a given finite Borel cover for an invariant probability measure, and then proved the equivalence of them, which answers Question 2.10 in <<combinatorial independence in measurable dynamics>> from http://arxiv.org/abs/0705.3424v1 by Kerr and Li. To prove the equivalence,

1. Besides the standard machinery developed by Ornstein and Weiss, it becomes an inevitable tool the orbital theory of dynamical systems built by Connes, Danilenko, Dye, Feldman, Moore, Weiss, Zimmer and other mathematicians.
2. As a bridge, we have to study the local entropy theory in the relative setting of given a factor map between dynamical systems. The result of the local variational principle concerning relative entropy has been published on **Ergodic Theory and Dynamical Systems**, 26 (2006), no. 1, 219-245; the paper has been cited by many mathematicians, including D. J. Feng, Glasner, H. Y. Hu, Kerr, H. F. Li, Shapira and Y. F. Yi, and it was listed by the official website of **Ergodic Theory and Dynamical Systems** as one of the top 10 most-read articles in 2007. In the proving of the result, it seems very difficult to generalize to the relative case the key combinatorial lemma in previous works, some new method is needed; the equivalence of two kind of entropy for $\mathbb{Z}$-actions in the absolute case was first pointed out by us in the paper, and it will play an important role in the building of the local entropy theory for a countable discrete amenable group action. As the applications we made a systematical study of the local theory of relative entropy in **Israel Journal of Mathematics**, 158 (2007), 249-284; in the process, we answered affirmatively a question raised by Park and Siemaszko by obtaining naturally the relative topological Pinsker factor (opposite to the Pinsker σ-algebra in ergodic theory).

Part 3. the relationship between entropy and chaos.

In this part, we studied the complexity of a system from the viewpoint of entropy and chaos in the relative setting. Precisely, we proved: the positivity of relative entropy implies the existence of proper asymptotic pairs on fibers and relative Mycielski chaos; moreover, the supremum of entropy of scrambled subsets on fibers is just relative entropy of the factor map. These results have been published on **Ergodic Theory and Dynamical Systems**, 27 (2007), no. 4, 1349-1371 and **Journal of London Mathematical Society**, 73 (2006), no. 1, 157-172, and the referee commented these results with “significantly deeper than the corresponding absolute statements”, “opens a new view on topological entropy”, “the absolute version of Theorem 4.4 … is striking”.

Besides these, during the PhD study and the one-year period after my graduation, with my cooperators I also carried out the study related to the following topics:

1. discussed the complexity of a system by considering the entropy of subsets. An open problem for a long time is whether each dynamical system with finite entropy admits a symbolic extension. Many mathematicians have studied it, including Boyle, Downarowicz, D. Fiebig, U.

Fiebig, Glasner, Newhouse, Serafin and Weiss. The question was solved completely by Boyle and Downarowicz till 2004. They proved that a dynamical system admits a principal symbolic extension if and only if it is asymptotically h-expansive. Our results gave a different and completely new dynamical characterization of asymptotical h-expansiveness: a dynamical system is asymptotically h-expansive if and only if it is hereditarily uniformly lowerable. Moreover, each dynamical system with finite entropy must be lowerable. These results have been accepted by **Ergodic Theory and Dynamical Systems**, and are commented by the referee with "well-worth publishing".

2. introduced the (global and local) notion of measure-theoretic pressure, gave its equivalent characterization and then established the (global and local) variational relationship between it and topological pressure. These results have been accepted by **Discrete and Continuous Dynamical Systems, Ser. A**.
3. studied some questions related to the local variational principle concerning entropy. We found that these questions had deep relationships with many mathematical branches.
4. The extensive existence and the possibility of being observed of chaotic behaviour. We aim to study the complexity of a dynamical system, such as entropy, chaos and so on, in the most general setting. We found that, under some necessary assumptions, chaotic behaviour of a system exists always and can be observed from many different viewpoints.

Key words: local entropy, amenable group, local variational principle, relative entropy, relative chaos